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Maxwell's equations in spaces with non-metricity and torsion

M A Vandyck†

Physics Department, University College Cork, Cork City, Ireland
and
Physics Department, Cork Regional Technical College, Bishopstown, County Cork, Ireland

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Abstract. The formulation of Maxwell's equations using exterior differentiation is compared to that involving covariant differentiation. These two formulations are known to be equivalent in a space with a Riemannian connection, and a *necessary and sufficient* condition is established here for this equivalence to be maintained in the case where the connection is of the most general type, namely a connection with, in general, torsion and non-metricity, in addition to curvature.

1. Introduction

It is well known [1] that, in Cartesian coordinates in Minkowski space, Maxwell's equations may be written as

$$F_{[\mu\nu,\rho]} = 0 \qquad G^{\mu\nu}{}_{,\nu} = -lJ^{\mu} \tag{1.1}$$

in terms of the components $F_{\mu\nu}$ and $G^{\mu\nu}$ of two antisymmetric tensors F and G. In (1.1), l is a constant depending on the system of units, and the comma denotes partial differentiation. Alternatively, using Hodge dualization, (1.1) becomes [2]

$$\mathrm{d}F = 0 \qquad \mathrm{d}\,{}^*\tilde{G} = -l\,{}^*\tilde{J} \tag{1.2}$$

$$2F \equiv F_{\mu\nu}\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \qquad 2\tilde{G} \equiv G_{\mu\nu}\,\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \qquad \tilde{J} \equiv J_{\mu}\,\mathrm{d}x^{\mu} \tag{1.3}$$

where the indices of $G^{\mu\nu}$ and J^{μ} have been lowered (by the Minkowski metric), which is emphasized by the notation $\tilde{G}$ and $\tilde{J}$.

When attempting to apply Maxwell's equations in a more general C^{∞} manifold $\mathcal{M}$, one notes that (1.2) is mathematically meaningful and therefore does not require any modification, whereas (1.1) is usually generalized by the 'comma going to semi-colon' rule [3], namely by replacing the partial differentiation of the components $F_{\mu\nu}$ and $G^{\mu\nu}$ by the covariant differentiation of the tensors $F \equiv F_{\mu\nu}\,\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu}$ and $G \equiv G^{\mu\nu}(\partial/\partial x^{\mu}) \otimes (\partial/\partial x^{\nu})$ as

$$F_{[\mu\nu;\rho]} = 0 \qquad G^{\mu\nu}{}_{;\nu} = -lJ^{\mu}. \tag{1.4}$$

Therefore, in a general C^{∞} manifold (with connection), both forms (1.4) and (1.2) are meaningful and have the property that they reduce to (1.1) in Cartesian coordinates in Minkowski space.

† Research associate of the Dublin Institute for Advanced Studies.

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However, in general, it is clear that (1.4) depends on the choice of the connection, whereas (1.2) is independent of the connection. Consequently, it is impossible that (1.2) and (1.4) should be equivalent in general. Moreover, a *sufficient* condition for (1.2) and (1.4) to be equivalent is that the connection be Riemannian [3]. The question arises, thus, to find a *necessary* and sufficient condition for the equivalence to hold.

In this work, we shall consider the most general connection, which exhibits curvature, torsion, and non-metricity, and establish that (1.2) and (1.4) are equivalent if and only if the torsion vanishes and the non-metricity is traceless. (Renewed interest has recently been shown in connection with non-metricity [4, 5].)

In section 2, we shall express the exterior-calculus formulation (1.2) of Maxwell's equations in components in an arbitrary frame $\{\underset{\to}{e}^{(\mu)}\}$ in the cotangent space $T^*\mathcal{M}$ to the manifold $\mathcal{M}$. Then we shall, in section 3, develop the fundamental equations of the general connections with curvature, torsion, and non-metricity. In section 4, we shall 'translate' the exterior-calculus form (1.2) in covariant-derivative language, which will enable us to compare the results with the covariant-derivative form (1.4) of Maxwell's equations, and yield the desired necessary and sufficient condition for the equivalence of (1.2) and (1.4). Finally, in section 5, we shall solve the problem of the electrostatic field created by a point charge in a space with non-traceless non-metricity. This will provide a simple example where the difference between the two formulations of Maxwell's equations comes to light.

It is important to emphasize that, to solve Maxwell's equations in either of the forms (1.2) or (1.4), one needs to adopt a 'constitutive relation' linking F to G. None of the considerations made hereafter, in sections 2–4, will involve this constitutive relation. Only in section 5, where we shall solve a specific example, will the constitutive equation of the vacuum be employed, namely

$$\tilde{G} = kF \qquad \text{(vacuum)} \tag{1.5}$$

where k is a constant depending on the system of units.

Remark. The inhomogeneous Maxwell equation given in (1.4) is not the only conceivable generalization† of the inhomogeneous equation of (1.1). For instance, one might consider

$$g^{\mu\alpha} g^{\nu\beta} G_{\alpha\beta;\nu} = -lJ^{\mu} \tag{1.6}$$

which is distinct from the inhomogeneous equation of (1.4) when the connection is not metric-compatible.

There is, however, no difficulty in relating (1.6) to the inhomogeneous equation of (1.4). For the sake of clarity, the main body of this article will deal exclusively with (1.2) and (1.4), and the 'translation' of the results in terms of the form (1.6) will be performed in the appendix.

2. Exterior-calculus formulation in components

Let F and $\tilde{G}$ be the electro-magnetic two-forms of (1.3), expressed in the arbitrary basis $\{\underset{\to}{e}^{(\mu)}\}$ of $T^*\mathcal{M}$:

$$2F \equiv F_{\mu\nu}\underset{\to}{e}^{(\mu)} \wedge \underset{\to}{e}^{(\nu)} \qquad 2\tilde{G} \equiv G_{\mu\nu}\underset{\to}{e}^{(\mu)} \wedge \underset{\to}{e}^{(\nu)}. \tag{2.1}$$

The frame $\{\underset{\to}{e}^{(\mu)}\}$ is characterized by its commutation coefficients $D^{\mu}{}_{\alpha\beta}$, defined by

$$2\mathrm{d}\underset{\to}{e}^{(\mu)} = -D^{\mu}{}_{\alpha\beta}\underset{\to}{e}^{(\alpha)} \wedge \underset{\to}{e}^{(\beta)}. \tag{2.2}$$

† The author would like to thank the referee for this remark.

As the terminology suggests, they are related to the commutator of the basic vectors $\{e_{(\mu)}\}$, dual to $\{\underset{\rightarrow}{e}^{(\mu)}\}$, by

$$[e_{(\mu)}, e_{(\nu)}] = D^{\alpha}{}_{\mu\nu} e_{(\alpha)}. \tag{2.3}$$

The homogeneous Maxwell equation from (1.2) may then be written as

$$0 = 2\mathrm{d}F = \mathrm{d}\{F_{\mu\nu}\underset{\rightarrow}{e}^{(\mu)} \wedge \underset{\rightarrow}{e}^{(\nu)}\} \tag{2.4}$$

$$= (\mathrm{d}F_{\mu\nu}) \wedge \underset{\rightarrow}{e}^{(\mu)} \wedge \underset{\rightarrow}{e}^{(\nu)} + F_{\mu\nu}\,\mathrm{d}(\underset{\rightarrow}{e}^{(\mu)} \wedge \underset{\rightarrow}{e}^{(\nu)}) \tag{2.5}$$

$$= \{e_{(\rho)}(F_{\mu\nu}) - F_{\alpha\rho}D^{\alpha}{}_{\mu\nu}\}\underset{\rightarrow}{e}^{(\rho)} \wedge \underset{\rightarrow}{e}^{(\mu)} \wedge \underset{\rightarrow}{e}^{(\nu)} \tag{2.6}$$

where we have employed (2.2). Owing to the total antisymmetry of $\underset{\rightarrow}{e}^{(\rho)} \wedge \underset{\rightarrow}{e}^{(\mu)} \wedge \underset{\rightarrow}{e}^{(\nu)}$, (2.6) is equivalent to

$$0 = e_{[(\rho)}(F_{\mu\nu]}) - F_{\alpha[\rho}D^{\alpha}{}_{\mu\nu]}. \tag{2.7}$$

Furthermore, to treat the inhomogeneous Maxwell equation from (1.2), one needs the definition of the Hodge dual. We adopt here the following convention, for an arbitrary q-form ${}_q\alpha$, $0 \leqslant q \leqslant 4$, in four dimensions,

$$\overset{*}{{}_q}\alpha \equiv \frac{1}{q!(4-q)!}\sqrt{|g|}\,{}_q\alpha^{\mu_1\ldots\mu_q}\epsilon_{\mu_1\ldots\mu_4}\underset{\rightarrow}{e}^{(\mu_{q+1})} \wedge \ldots \wedge \underset{\rightarrow}{e}^{(\mu_4)} \tag{2.8}$$

where $|g|$ and $\epsilon_{\mu_1\ldots\mu_4}$ denote the absolute value of the determinant of the matrix $g_{\mu\nu}$ of the covariant components of the metric, and the totally antisymmetric symbol in four dimensions with $\epsilon_{1234} = +1$, respectively.

The inhomogeneous equation from (1.2) is now easily obtained in components. One first evaluates the Hodge duals ${}^*\tilde{G}$ and ${}^*\tilde{J}$ as being

$$4\,{}^*\tilde{G} = \sqrt{|g|}G^{\alpha\beta}\epsilon_{\alpha\beta\mu\nu}\underset{\rightarrow}{e}^{(\mu)} \wedge \underset{\rightarrow}{e}^{(\nu)} \tag{2.9}$$

$$6\,{}^*\tilde{J} = \sqrt{|g|}J^{\alpha}\epsilon_{\alpha\rho\mu\nu}\underset{\rightarrow}{e}^{(\rho)} \wedge \underset{\rightarrow}{e}^{(\mu)} \wedge \underset{\rightarrow}{e}^{(\nu)}. \tag{2.10}$$

After a calculation similar to that leading to (2.7), the inhomogeneous equation (1.2) becomes

$$\frac{1}{\sqrt{|g|}}\epsilon_{\alpha\beta[\mu\nu}e_{(\rho)]}(\sqrt{|g|}G^{\alpha\beta}) - \epsilon_{\alpha\beta\gamma[\rho}D^{\gamma}{}_{\mu\nu]}G^{\alpha\beta} = -\tfrac{2}{3}lJ^{\alpha}\epsilon_{\alpha\rho\mu\nu}. \tag{2.11}$$

In order to simplify (2.11), we contract it with $\epsilon^{\theta\rho\mu\nu}$ and employ the combinatorial identities

$$\epsilon_{\mu\alpha\beta\gamma}\epsilon^{\nu\alpha\beta\gamma} = 3!\delta^{\nu}_{\mu} \qquad \epsilon_{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\alpha\beta} = 2!2!\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]} \qquad \epsilon_{\mu\nu\rho\alpha}\epsilon^{\sigma\tau\phi\alpha} = 3!\delta^{\sigma}_{[\mu}\delta^{\tau}_{\nu}\delta^{\phi}_{\rho]} \tag{2.12}$$

to yield

$$\frac{1}{\sqrt{|g|}}e_{(\nu)}(\sqrt{|g|}G^{\theta\nu}) + D^{\alpha}{}_{\alpha\beta}G^{\theta\beta} - \tfrac{1}{2}D^{\theta}{}_{\alpha\beta}G^{\alpha\beta} = -lJ^{\theta}. \tag{2.13}$$

The conclusion that we reach is that Maxwell's equations, in the language of differential forms, read, in coordinate-free and component formulation,

$$\mathrm{d}F = 0 \Leftrightarrow e_{[(\rho)}(F_{\mu\nu]}) - F_{\alpha[\rho}D^{\alpha}{}_{\mu\nu]} = 0 \tag{2.14}$$

$$\mathrm{d}\,{}^*\tilde{G} = -l\,{}^*\tilde{J} \Leftrightarrow \frac{1}{\sqrt{|g|}}e_{(\nu)}(\sqrt{|g|}G^{\mu\nu}) + D^{\alpha}{}_{\alpha\beta}G^{\mu\beta} - \tfrac{1}{2}D^{\mu}{}_{\alpha\beta}G^{\alpha\beta} = -lJ^{\mu}. \tag{2.15}$$

These results hold in full generality for a C^{∞} manifold $\mathcal{M}$. Incidentally, we also see from (2.14) and (2.15) that, for Minkowski space in Cartesian coordinates, the commutation coefficients D of the basis $e_{(\mu)} \equiv \partial/\partial x^{\mu}$ vanish, and the metric has determinant -1, so that (2.14) and (2.15), coming from exterior differentiation, reproduce, in that very special case, the correct Maxwell equations (1.1).

To prepare the comparison, in full generality, between (2.14), (2.15) and the covariant-derivative form (1.4), we need to recall the fundamental equations of connection theory, in the presence of curvature, torsion, and non-metricity. This will be done in the next section. Detailed proofs of the relevant theorems will, however, not be given below, since they are similar to those of the torsion-free, metric-compatible case, available in text books, for instance in [6].

3. Connections with torsion and non-metricity

In general, a connection Γ, expressed in components by

$$\nabla_{\boldsymbol{e}_{(\mu)}} \boldsymbol{e}_{(\nu)} = \Gamma^{\alpha}{}_{\nu\mu} \boldsymbol{e}_{(\alpha)} \tag{3.1}$$

where ∇ denotes the operator of covariant differentiation, exhibits curvature R and torsion T. The latter read

$$T(\boldsymbol{u}, \boldsymbol{v}) \equiv \nabla_{\boldsymbol{u}} \boldsymbol{v} - \nabla_{\boldsymbol{v}} \boldsymbol{u} - [\boldsymbol{u}, \boldsymbol{v}] = -T(\boldsymbol{v}, \boldsymbol{u}) \tag{3.2}$$

$$R(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \equiv (\nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{v}} - \nabla_{\boldsymbol{v}} \nabla_{\boldsymbol{u}} - \nabla_{[\boldsymbol{u}, \boldsymbol{v}]}) \boldsymbol{w} = -R(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}) \tag{3.3}$$

in which $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are arbitrary C^{∞} vector fields. (In what follows, the curvature will play no role, and therefore we shall not pursue further the treatment of R.) In order to obtain the component expression of the torsion, we substitute $\boldsymbol{u}$ and $\boldsymbol{v}$ in (3.2) by the basic vectors $\boldsymbol{e}_{(\alpha)}$ and $\boldsymbol{e}_{(\beta)}$, and find

$$T^{\mu}{}_{\alpha\beta} \equiv \underset{\rightarrow}{e}^{(\mu)} \{ T(\boldsymbol{e}_{(\alpha)}, \boldsymbol{e}_{(\beta)}) \} = 2\Gamma^{\mu}{}_{[\beta\alpha]} - D^{\mu}{}_{\alpha\beta} = -T^{\mu}{}_{\beta\alpha} \tag{3.4}$$

where (2.3) has been employed.

Moreover, from the metric tensor g and the covariant derivative ∇, one may introduce the non-metricity tensor H as

$$H(\boldsymbol{v}, \boldsymbol{x}, \boldsymbol{y}) = (\nabla_{\boldsymbol{v}} g)(\boldsymbol{x}, \boldsymbol{y}) = H(\boldsymbol{v}, \boldsymbol{y}, \boldsymbol{x}) \tag{3.5}$$

for all $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{v}$. By virtue of (3.1), this becomes, in components,

$$H_{\mu\alpha\beta} \equiv H(\boldsymbol{e}_{(\mu)}, \boldsymbol{e}_{(\alpha)}, \boldsymbol{e}_{(\beta)}) = \boldsymbol{e}_{(\mu)}(g_{\alpha\beta}) - \Gamma_{\beta\alpha\mu} - \Gamma_{\alpha\beta\mu} = H_{\mu\beta\alpha} \tag{3.6}$$

where we put

$$\Gamma_{\alpha\beta\gamma} \equiv g_{\alpha\kappa} \Gamma^{\kappa}{}_{\beta\gamma}. \tag{3.7}$$

The important point, for our purposes, is that, from the component expressions (3.4) and (3.6) of the torsion and the non-metricity, it is possible to determine the connection coefficients $\Gamma_{\alpha\beta\gamma}$ as

$$\Gamma_{\alpha\beta\gamma} = \langle\alpha\beta\gamma\rangle + Q_{\alpha\beta\gamma} - K_{\alpha\beta\gamma} \tag{3.8}$$

where we introduced the notation

$$\langle\alpha\beta\gamma\rangle \equiv [\alpha\beta\gamma] + C_{\alpha\beta\gamma} \qquad \text{(Levi–Civita or Riemannian connection)} \tag{3.9}$$

$$2[\alpha\beta\gamma] \equiv \boldsymbol{e}_{(\gamma)}(g_{\alpha\beta}) + \boldsymbol{e}_{(\beta)}(g_{\alpha\gamma}) - \boldsymbol{e}_{(\alpha)}(g_{\beta\gamma}) = 2[\alpha\gamma\beta] \qquad \text{(Christoffel symbol of the 1st kind)} \tag{3.10}$$

$$2C_{\alpha\beta\gamma} \equiv D_{\gamma\alpha\beta} + D_{\beta\alpha\gamma} - D_{\alpha\beta\gamma} = -2C_{\beta\alpha\gamma} \qquad \text{(anholonomic connection)} \tag{3.11}$$

$$2Q_{\alpha\beta\gamma} \equiv T_{\gamma\alpha\beta} + T_{\beta\alpha\gamma} - T_{\alpha\beta\gamma} = -2Q_{\beta\alpha\gamma} \qquad \text{(contorsion tensor)} \tag{3.12}$$

$$2K_{\alpha\beta\gamma} \equiv H_{\gamma\alpha\beta} + H_{\beta\alpha\gamma} - H_{\alpha\beta\gamma} = +2K_{\alpha\gamma\beta} \qquad \text{(metric-incompatibility tensor).} \tag{3.13}$$

As a consequence of (3.8)–(3.13), when the basis $\{\boldsymbol{e}_{(\mu)}\}$ has been chosen (so that $D^{\mu}{}_{\alpha\beta}$ is known), there exists a unique connection Γ admitting a given metric g, torsion T, and non-metricity H.

Alternatively to (3.8), one may raise the first index of the connection coefficients (using the metric), and obtain

$$\Gamma^{\alpha}{}_{\beta\gamma} = \langle^{\alpha}{}_{\beta\gamma}\rangle + Q^{\alpha}{}_{\beta\gamma} - K^{\alpha}{}_{\beta\gamma} \tag{3.14}$$

where, in the quantity $\langle^{\alpha}{}_{\beta\gamma}\rangle$, the contribution $g^{\alpha\kappa}[\kappa\beta\gamma]$, which arises from (3.9), is denoted by $\{^{\alpha}{}_{\beta\gamma}\}$ and called the 'Christoffel symbol of the second kind'. It will be necessary in the following section.

In the special case where $H = 0$, so that $K = 0$, the connection is said to be 'metric-compatible'. If, in the metric-compatible case, the connection admits torsion, namely if $T \neq 0 \neq Q$, the connection is said to be of the 'Riemann–Cartan' type, whereas if the torsion vanishes, the connection is said to be 'Levi–Civita' (or 'Riemannian'), which is the well known connection used in general relativity. Riemann–Cartan connections appear, for instance, in the Poincaré gauge-field theory [7] or in supergravity [8]. The special (torsion-free) non-metric-compatible connection characterized by

$$H_{\alpha\beta\gamma} \equiv 2A_{\alpha}g_{\beta\gamma} \tag{3.15}$$

where A is a covariant vector field, is employed in Weyl's theory [9] of gravity. (In this context, A is usually called the 'Weyl vector'.) See also [4, 5] for recent developments relating non-metric-compatible connections to dilaton gravity.

We shall not develop here the geometrical interpretation of the torsion and the non-metricity in any detail. (The reader is referred to [10] for a more comprehensive treatment.) It is sufficient for our purposes to recall that the non-metricity expresses how the scalar product $\boldsymbol{v}\cdot\boldsymbol{w}$ of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ varies when $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel-transported along a vector field $\boldsymbol{x}$. More precisely,

$$\boldsymbol{x}[\boldsymbol{v}\cdot\boldsymbol{w}] \equiv \boldsymbol{x}[g(\boldsymbol{v},\boldsymbol{w})] = H(\boldsymbol{x},\boldsymbol{v},\boldsymbol{w}) = H_{\alpha\mu\nu}x^{\alpha}v^{\mu}w^{\nu} \tag{3.16}$$

where $\boldsymbol{x}[f]$ denotes the rate of change of a function f along $\boldsymbol{x}$. Furthermore, if the non-metricity $H_{\alpha\mu\nu}$ is decomposed into its trace in the last two indices and the traceless remainder as

$$H_{\alpha\mu\nu} = [H_{\alpha\mu\nu} - \tfrac{1}{4}(g^{\theta\lambda}H_{\alpha\theta\lambda})g_{\mu\nu}] + \tfrac{1}{4}(g^{\theta\lambda}H_{\alpha\theta\lambda})g_{\mu\nu} \tag{3.17}$$

$$\equiv {}^{0}H_{\alpha\mu\nu} + H_{\alpha}g_{\mu\nu} \tag{3.18}$$

$$g^{\mu\nu}\,{}^{0}H_{\alpha\mu\nu} = 0 \qquad 4H_{\alpha} \equiv g^{\theta\lambda}H_{\alpha\theta\lambda} \tag{3.19}$$

it is easy to establish that two parallel-transported vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ experience a shear coming from the traceless part ${}^{0}H_{\alpha\mu\nu}$ and an expansion coming from the trace H_{α}, whereas they undergo a Lorentz transformation in the absence of both ${}^{0}H_{\alpha\mu\nu}$ and H_{α}. In particular, in a Weyl space, characterized by (3.15), the traceless part ${}^{0}H_{\alpha\mu\nu}$ vanishes, and the trace part H_{α} is given by $H_{\alpha} = 2A_{\alpha}$ in terms of the Weyl vector A.

We are now ready to investigate the covariant-derivative formulation (1.4) of Maxwell's equations and its relationship with the exterior-differentiation form (2.14) and (2.15).

4. Exterior-calculus formulation in covariant-derivative language

If one considers Maxwell's homogeneous equation (2.14) in components, the contribution $\boldsymbol{e}_{[(\rho)}(F_{\mu\nu]})$ is reminiscent of the term $F_{[\mu\nu;\rho]}$, which appears in the form (1.4) involving covariant derivatives. Let us therefore calculate $F_{\mu\nu;\rho}$ in terms of the connection coefficients.

By virtue of definition (3.1) of the connection coefficients, one has

$$F_{\mu\nu;\rho} = e_{(\rho)}(F_{\mu\nu}) - F_{\alpha\nu}\Gamma^{\alpha}{}_{\mu\rho} - F_{\mu\alpha}\Gamma^{\alpha}{}_{\nu\rho} \tag{4.1}$$

$$= e_{(\rho)}(F_{\mu\nu}) - F^{\alpha}{}_{\nu}\Gamma_{\alpha\mu\rho} + F^{\alpha}{}_{\mu}\Gamma_{\alpha\nu\rho} \tag{4.2}$$

where the antisymmetry of F has been exploited. Therefore, the antisymmetric part $F_{[\mu\nu;\rho]}$ reads

$$F_{[\mu\nu;\rho]} = e_{[\rho}(F_{\mu\nu]}) - F^{\alpha}{}_{[\nu}\Gamma_{|\alpha|\mu\rho]} + F^{\alpha}{}_{[\mu}\Gamma_{|\alpha|\nu\rho]} = e_{[\rho}(F_{\mu\nu]}) + 2F^{\alpha}{}_{[\mu}\Gamma_{|\alpha|\nu\rho]} \tag{4.3}$$

in which antisymmetrization is not applied to indices between vertical bars.

Furthermore, we saw in the previous section that, according to (3.8), the connection components $\Gamma_{\alpha\mu\nu}$ are given by the combination of the Riemannian connection $\langle\alpha\mu\nu\rangle$, the contorsion tensor $Q_{\alpha\mu\nu}$, and the metric-incompatibility tensor $K_{\alpha\mu\nu}$. After substituting this decomposition into (4.3), there follows

$$F_{[\mu\nu;\rho]} = F_{[\mu\nu!\rho]} + 2F^{\alpha}{}_{[\mu}Q_{|\alpha|\nu\rho]} - 2F^{\alpha}{}_{[\mu}K_{|\alpha|\nu\rho]} \tag{4.4}$$

in which we have adopted the 'exclamation mark' notation for the covariant derivative of a tensor with respect to the Riemannian part of the connection

$$F_{[\mu\nu!\rho]} = e_{[\rho}(F_{\mu\nu]}) + 2F^{\alpha}{}_{[\mu}\langle|\alpha|\nu\rho]\rangle. \tag{4.5}$$

Given that, as found in (3.13) of section 3, the metric-incompatibility tensor K is symmetric in its last two indices, the antisymmetrization involving K in (4.4) vanishes. In addition, the expression of the contorsion tensor Q in terms of the torsion T is known from (3.12). As a result, (4.4) simplifies as

$$F_{[\mu\nu;\rho]} = F_{[\mu\nu!\rho]} - F^{\alpha}{}_{[\mu}T_{|\alpha|\nu\rho]}. \tag{4.6}$$

What remains to be done is to evaluate the Riemannian part $F_{[\mu\nu!\rho]}$.

The Riemannian part $\langle\mu\nu\rho\rangle$ of the connection is the sum of the Christoffel symbol $[\mu\nu\rho]$ of the first kind and the anholonomic connection $C_{\mu\nu\rho}$, as seen in (3.9). The Christoffel symbol is symmetrical in its last two indices, so that the antisymmetrization appearing in (4.5) receives no contribution from it. Furthermore, when the anholonomic connection C is substituted into (4.5) by its value from (3.11) in terms of the commutation coefficients D, a simple calculation yields

$$F_{[\mu\nu!\rho]} = e_{[(\rho)}(F_{\mu\nu]}) - F_{\alpha[\rho}D^{\alpha}{}_{\mu\nu]}. \tag{4.7}$$

When (4.7) and (4.6) are compared with Maxwell's homogeneous equation (2.14) in the exterior-differential formulation, one obtains the following four equivalent expressions:

$$\mathrm{d}F = 0 \Leftrightarrow e_{[(\rho)}(F_{\mu\nu]}) - F_{\alpha[\rho}D^{\alpha}{}_{\mu\nu]} = 0 \tag{4.8}$$

$$\Leftrightarrow F_{[\mu\nu!\rho]} = 0 \tag{4.9}$$

$$\Leftrightarrow F_{[\mu\nu;\rho]} + F^{\alpha}{}_{[\mu}T_{|\alpha|\nu\rho]} = 0. \tag{4.10}$$

Let us consider now the inhomogeneous Maxwell equation (2.15). This equation may be simplified by exactly the same method as the one that we applied to the homogeneous equation, and therefore we shall not present here all the details of the calculation.

One begins by again using definition (3.1) of the connection to evaluate $G^{\mu\nu}{}_{;\nu}$ in terms of the connection coefficients $\Gamma^{\mu}{}_{\nu\rho}$, yielding

$$G^{\mu\nu}{}_{;\nu} = e_{(\nu)}(G^{\mu\nu}) + \Gamma^{\mu}{}_{\alpha\beta}G^{\alpha\beta} + \Gamma^{\alpha}{}_{\beta\alpha}G^{\mu\beta}. \tag{4.11}$$

The coefficients $\Gamma^{\mu}{}_{\nu\rho}$ are then substituted by their values in terms of the Riemannian connection $\langle^{\mu}{}_{\nu\rho}\rangle$, the contorsion tensor $Q^{\mu}{}_{\nu\rho}$, and the metric-incompatibility tensor $K^{\mu}{}_{\nu\rho}$, with the result

$$G^{\mu\nu}{}_{;\nu} = e_{(\nu)}(G^{\mu\nu}) + (\langle^{\mu}{}_{\alpha\beta}\rangle + Q^{\mu}{}_{\alpha\beta} - K^{\mu}{}_{\alpha\beta})G^{\alpha\beta} + (\langle^{\alpha}{}_{\beta\alpha}\rangle + Q^{\alpha}{}_{\beta\alpha} - K^{\alpha}{}_{\beta\alpha})G^{\mu\beta}. \tag{4.12}$$

After grouping together the terms involving the Riemannian connection $\langle^{\alpha}{}_{\beta\gamma}\rangle$, and exploiting the symmetry of the metric-incompatibility tensor K in its last two indices, (4.12) becomes

$$G^{\mu\nu}{}_{;\nu} = G^{\mu\nu}{}_{!\nu} + Q^{\mu}{}_{\alpha\beta}G^{\alpha\beta} + (Q^{\alpha}{}_{\beta\alpha} - K^{\alpha}{}_{\beta\alpha})G^{\mu\beta} \tag{4.13}$$

$$G^{\mu\nu}{}_{!\nu} = \boldsymbol{e}_{(\nu)}(G^{\mu\nu}) + \langle^{\mu}{}_{\alpha\beta}\rangle G^{\alpha\beta} + \langle^{\alpha}{}_{\beta\alpha}\rangle G^{\mu\beta}. \tag{4.14}$$

We now replace in (4.13) the contorsion Q and the metric-incompatibility K by their expressions (3.12) and (3.13) in terms of the torsion T and the non-metricity H, with the consequence that

$$G^{\mu\nu}{}_{;\nu} = G^{\mu\nu}{}_{!\nu} - \tfrac{1}{2}T^{\mu}{}_{\alpha\beta}G^{\alpha\beta} + (T^{\alpha}{}_{\alpha\beta} - \tfrac{1}{2}H_{\beta}{}^{\alpha}{}_{\alpha})G^{\mu\beta}. \tag{4.15}$$

To evaluate the Riemannian contribution $G^{\mu\nu}{}_{!\nu}$ of (4.14), one recalls that the Riemannian connection $\langle^{\mu}{}_{\nu\rho}\rangle$ is the sum of the Christoffel symbol $\{^{\mu}{}_{\nu\rho}\}$ of the second kind and the anholonomic connection $C^{\mu}{}_{\nu\rho}$. The latter is known in terms of the commutation coefficients $D^{\mu}{}_{\nu\rho}$ by (3.11). After substitution of these relationships into (4.14), one finds

$$G^{\mu\nu}{}_{!\nu} = \boldsymbol{e}_{(\nu)}(G^{\mu\nu}) + \{^{\alpha}{}_{\beta\alpha}\}G^{\mu\beta} + D^{\alpha}{}_{\alpha\beta}G^{\mu\beta} - \tfrac{1}{2}D^{\mu}{}_{\alpha\beta}G^{\alpha\beta}. \tag{4.16}$$

Moreover, a simple calculation based on the properties of determinants yields [11]

$$\{^{\alpha}{}_{\beta\alpha}\} = \frac{1}{2}\frac{1}{|g|}\boldsymbol{e}_{(\beta)}(|g|) \tag{4.17}$$

which enables one to reformulate (4.16) as

$$G^{\mu\nu}{}_{!\nu} = \frac{1}{\sqrt{|g|}}\boldsymbol{e}_{(\nu)}(\sqrt{|g|}G^{\mu\nu}) + D^{\alpha}{}_{\alpha\beta}G^{\mu\beta} - \tfrac{1}{2}D^{\mu}{}_{\alpha\beta}G^{\alpha\beta}. \tag{4.18}$$

When (4.18) and (4.15) are compared with the inhomogeneous Maxwell equation (2.15), coming from exterior differentiation, one obtains the following four equivalent formulations:

$$\mathrm{d}\,{}^*\tilde{G} = -l\,{}^*\tilde{J} \quad\Leftrightarrow\quad \frac{1}{\sqrt{|g|}}\boldsymbol{e}_{(\nu)}(\sqrt{|g|}G^{\mu\nu}) + D^{\alpha}{}_{\alpha\beta}G^{\mu\beta} - \tfrac{1}{2}D^{\mu}{}_{\alpha\beta}G^{\alpha\beta} = -lJ^{\mu} \tag{4.19}$$

$$\Leftrightarrow G^{\mu\nu}{}_{!\nu} = -lJ^{\mu} \tag{4.20}$$

$$\Leftrightarrow G^{\mu\nu}{}_{;\nu} + (\tfrac{1}{2}H_{\beta}{}^{\alpha}{}_{\alpha} - T^{\alpha}{}_{\alpha\beta})G^{\mu\beta} + \tfrac{1}{2}T^{\mu}{}_{\alpha\beta}G^{\alpha\beta} = -lJ^{\mu}. \tag{4.21}$$

These expressions, together with (4.8)–(4.10), constitute the full set of Maxwell's equations in matter, as they arise from the formalism involving exterior differentiation. We are now in the position to study the relationship between these equations and those obtained, in (1.4), in covariant-derivative form.

What is obvious from (4.9) and (4.20) is that the exterior-calculus formulation is *not* equivalent to the covariant-derivative form (1.4), but rather to

$$F_{[\mu\nu!\rho]} = 0 \qquad G^{\mu\nu}{}_{!\nu} = -lJ^{\mu}. \tag{4.22}$$

In other words, the exterior-calculus formulation is equivalent to the covariant-derivative form *based on the Riemannian part* of the total connection Γ. It follows (trivially) that the two formulations agree if the connection Γ is Riemannian. This is well known [3]; what is more important is that the equivalence is maintained for a large class of connections Γ containing the Riemannian connections as special cases, as we shall now establish.

When the two versions (1.4) and (4.10) of Maxwell's homogeneous equation are compared to each other, one sees that the term by which they differ vanishes, for all fields F, if and only if the torsion T vanishes. Moreover, an analogous comparison between the two versions (1.4) and (4.21) of the inhomogeneous equation shows that both versions become identical if and only if the torsion vanishes, as well as the quantity $H_{\beta}{}^{\alpha}{}_{\alpha}$. The latter requirement means, in the terminology of section 3, that the non-metricity must be traceless.

The conclusion that we reach is thus that the vanishing of the torsion T and of the trace $4H_\mu \equiv H_\mu{}^\alpha{}_\alpha$ of the non-metricity is the necessary and sufficient condition for the two formulations of Maxwell's equations to be equivalent. This class of connections is larger than the class of Riemannian connections since it allows the additional freedom of the traceless part ${}^0H_{\mu\alpha\beta}$ of the non-metricity. When the necessary and sufficient condition is not satisfied, the two forms of Maxwell's equations are genuinely inequivalent. For instance, the two forms differ for a Weyl connection, defined by (3.15), and for a metric-compatible connection with torsion (employed, for instance, in supergravity [8]). To illustrate the inequivalence of the two forms, we shall investigate the example of the electrostatic field created by a point charge *in vacuo* in the following section.

5. Point charge and non-metricity

Let us consider the problem of determining the electrostatic field produced, *in vacuo*, by a point charge Q. The manifold $\mathcal{M}$ in which Q resides is assumed to possess a metric g given by

$$g = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\phi^2 - \mathrm{d}t^2. \tag{5.1}$$

Moreover, we also assume that the torsion vanishes and that the non-metricity takes the simple form

$$2H_{\mu\nu\rho} \equiv \boldsymbol{e}_{(\mu)}(h) g_{\nu\rho} \tag{5.2}$$

for a certain scalar field h. In other words, with the terminology of section 3, the manifold $\mathcal{M}$ under consideration is a special type of Weyl space, where the Weyl vector A is given by

$$4A_\mu = \boldsymbol{e}_{(\mu)}(h). \tag{5.3}$$

The justification for these choices is that (5.1) is the Minkowski metric. Moreover, when h is constant, the non-metricity vanishes, which, together with the vanishing of the torsion, implies that $\mathcal{M}$ is then Minkowski space. Thus, we are dealing here with a very simple generalization of Minkowski space, which reduces to Minkowski space when h is constant. Furthermore, by virtue of (5.2), the non-metricity is trace-free if and only if h is constant. Consequently, according to the necessary and sufficient condition of section 4, the two forms (1.2) and (1.4) of Maxwell's equations will yield equivalent results if and only if h is constant, as we shall see.

For all the calculations that follow, we shall use the cotangent-space orthonormal frame $\underset{\rightarrow}{e}^{(\hat{\mu})}$ defined by

$$\underset{\rightarrow}{e}^{(\hat{\mu})} \equiv \begin{pmatrix} \mathrm{d}r \\ r\,\mathrm{d}\theta \\ r\sin\theta\,\mathrm{d}\phi \\ \mathrm{d}t \end{pmatrix} \tag{5.4}$$

so that the metric (5.1) becomes

$$g = \eta_{\mu\nu} \underset{\rightarrow}{e}^{(\hat{\mu})} \otimes \underset{\rightarrow}{e}^{(\hat{\nu})} \qquad \eta_{\mu\nu} \equiv \mathrm{diag}(1, 1, 1, -1). \tag{5.5}$$

(Indices referring to an orthonormal basis are indicated by a caret.)

To describe the electrostatic field of a point charge Q located at $r = 0$, we take, as a source, the four-current density $J^{\hat{\mu}}$ as

$$J^{\hat{4}} = \frac{Q}{4\pi r^2}\delta(r) \qquad J^{\hat{1}} = J^{\hat{2}} = J^{\hat{3}} = 0 \tag{5.6}$$

where δ denotes Dirac's distribution. Furthermore, we assume that the only non-vanishing components of $G^{\hat{\mu}\hat{\nu}}$ are

$$G^{\hat{1}\hat{4}} = -G^{\hat{4}\hat{1}} \equiv D(r) \tag{5.7}$$

where D is a function of r only, interpreted as the radial component of the electric displacement $\boldsymbol{D}$. We also adopt the constitutive equation (1.5) of the vacuum, which determines F in terms of G. (The components $F^{\hat{1}\hat{4}} = -F^{\hat{4}\hat{1}}$ are interpreted as the radial component of electric field $\boldsymbol{E}$.)

Consider first the exterior-calculus form (2.14) and (2.15) of Maxwell's equations. There is no difficulty in evaluating the commutation coefficients $D^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}}$ of frame (5.4) by the application of (2.2), and to check that, for the fields (5.6) and (5.7), the only non-trivial Maxwell equation reads

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^2 D) = \frac{lQ}{4\pi}\delta(r). \tag{5.8}$$

After integration, this yields

$$D(r) = \frac{lQ}{4\pi r^2}\mathcal{H}(r) + \frac{C}{r^2} \tag{5.9}$$

where $\mathcal{H}$ denotes Heaviside's unit-step function, and C is an arbitrary constant. For physical reasons, we put C equal to zero, so as to ensure that the electric displacement D vanishes when $Q = 0$. The final answer thus becomes the well known Coulomb field

$$D(r) = \frac{lQ}{4\pi r^2} = kE(r) \qquad r > 0 \tag{5.10}$$

in which the the electric field E has been obtained from D by the constitutive equation (1.5) of the vacuum. (In the Gaussian system, k and l have the value 1 and 4π, respectively.)

The final result (5.10), which follows from the exterior-calculus form of Maxwell's equations, is valid for the Weyl space $\mathcal{M}$ in which the charge resides. In particular, if the non-metricity function h appearing in (5.2) is constant, (5.10) applies to Minkowski space. As mentioned in the introduction, the exterior-calculus form of Maxwell's equations is independent of the connection, which is manifest in (5.10) since E and D are independent of h. This will not be the case for the covariant-derivative form (1.4) of Maxwell's equations, as we shall now see.

To express (1.4), one may, but need not, calculate the connection using (3.8)–(3.13), so as to be able to evaluate the covariant derivatives present in (1.4). It is simpler to compare (1.4) with the set (4.10) and (4.21), which we have already analysed above. In our special case, where the torsion vanishes, the homogeneous equations are the same, and the inhomogeneous equations only differ by the term $\frac{1}{2}H_{\beta}{}^{\alpha}{}_{\alpha}G^{\mu\beta}$. As above, we use the frame (5.4) and the non-metricity (5.2), with a function h depending on r. The only non-trivial Maxwell equation, which corresponds to (5.8), reads

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^2 D) - (r^2 D)\frac{\mathrm{d}h}{\mathrm{d}r} = \frac{lQ}{4\pi}\delta(r). \tag{5.11}$$

It is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^2 D\mathrm{e}^{-h}) = \frac{lQ}{4\pi}\mathrm{e}^{-h}\delta(r) \tag{5.12}$$

and integrates as

$$D(r) = \frac{lQ}{4\pi r^2}\mathcal{H}(r)\mathrm{e}^{(h(r)-h(0))} + \frac{C}{r^2}\mathrm{e}^{h(r)} \tag{5.13}$$

where C is an arbitrary constant. For the same reason as in (5.9), we put C equal to zero, and find

$$D(r) = \frac{lQ}{4\pi r^2}\mathrm{e}^{(h(r)-h(0))} = kE(r) \qquad r > 0. \tag{5.14}$$

The final result (5.14) does now depend on h, unless h is constant, in contrast to what was the case in (5.10). We have thus exhibited an example where the outcomes of exterior-calculus form (1.2) and of the covariant-derivative form (1.4) of Maxwell's equations are different. The difference arises, in this example, from a non-metricity which is not traceless, unless the function h in (5.14) is constant. This is in keeping with the general necessary and sufficient condition of equivalence between the exterior-calculus form and the covariant-derivative form, established in section 4. In particular, when h is constant, the Weyl space $\mathcal{M}$ degenerates to Minkowski space, and the results (5.10) and (5.14) both coincide with the well known Coulomb field.

6. Conclusion

In this article, we considered the exterior-calculus form (1.2) and the covariant-derivative form (1.4) of Maxwell's equations, and we investigated under what conditions these forms are equivalent. A known *sufficient* condition [3] is that spacetime possesses a Riemannian connection. We established here that the *necessary* and sufficient condition is that the torsion and the trace of the non-metricity vanish. (This contains the Riemannian connection as a special case.) In other words, the two forms of Maxwell's equations are *inequivalent* in spaces admitting either torsion or non-metricity with trace (or both).

We then illustrated this construction by studying the problem of determining the vacuum electrostatic field produced by a point charge Q residing at the origin of a special Weyl space $\mathcal{M}$. This space differs from Minkowski space by the presence of non-traceless non-metricity, which is determined by a function h according to (5.2). When h is constant, $\mathcal{M}$ reduces to Minkowski space. In $\mathcal{M}$, the exterior-calculus form (1.2) of Maxwell's equations yields the electrostatic field (5.10), whereas the covariant-derivative form (1.4) yields the field (5.14). In accordance with the necessary and sufficient condition established in section 4, these fields are genuinely inequivalent, unless h is constant.

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Appendix

It is a simple matter to relate the alternative inhomogeneous equation (1.6) to the inhomogeneous equation of (1.4). To this end, one establishes first that the covariant derivative of the contravariant metric is given by

$$g^{\alpha\beta}{}_{;\xi} = -H_{\xi}{}^{\alpha\beta} \tag{A.1}$$

which follows from definition (3.5) of the non-metricity H and the Leibniz rule applied to the right-hand side of

$$0 = (g^{\mu\alpha} g_{\alpha\nu})_{;\xi}. \tag{A.2}$$

Then, the Leibniz rule is used once more, together with (A.1), to evaluate $G^{\mu\nu}{}_{;\nu}$ as being, after an elementary treatment,

$$G^{\mu\nu}{}_{;\nu} = (g^{\mu\alpha} g^{\nu\beta} G_{\alpha\beta})_{;\nu} \tag{A.3}$$

$$= g^{\mu\alpha} g^{\nu\beta} G_{\alpha\beta;\nu} + H_{\alpha\beta}{}^{\mu} G^{\alpha\beta} - H^{\alpha}{}_{\alpha\beta} G^{\mu\beta}. \tag{A.4}$$

The relationship (A.4) enables one to re-express the results obtained for the inhomogeneous Maxwell equation of (1.4), involving $G^{\mu\nu}{}_{;\nu}$, in terms of the quantity $g^{\mu\alpha} g^{\nu\beta} G_{\alpha\beta;\nu}$ which appears in the alternative equation (1.6), and hence to re-interpret our conclusions in terms of this alternative equation.

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